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The space of maps from a real projective space to a toric variety

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Abstract

The main purpose of this note is consider the homotopy type of the space of algebraic maps from a real projective space to a projective smooth toric variety as in [14]. The main result of this paper (Theorem 1.1) is also regarded as one of generalizations of the previous work of the second and third authors [19].

An irreducible normal algebraic variety X (over $\mathbb{C}$) is called *a toric variety* if it has an algebraic action of algebraic torus $\mathbb{T}^r=(\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}^r\cdot *$ of some point $*\in X$ is dense in X and isomorphic to $\mathbb{T}^r$. A finite correction Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called *a fan* if every face of element of Σ is belongs to Σ and the intersection of any two elements of Σ is a face of each. It is known that A toric variety X is completely characterized up to isomorphism by its fan Σ, and we denote by X_Σ the corresponding toric variety. For an n dimensional lattice polytope P, we denote by Σ_P *the normal fan* of P in $\mathbb{R}^n$. It is known that the toric variety X_Σ is projective if and only if $\Sigma=\Sigma_P$ for some n dimensional lattice polytope P in $\mathbb{R}^n$.

We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and for $f_1,\cdots,f_s\in\mathbb{C}[z_1,\cdots,z_r]$, let $V(f_1,\cdots,f_s)$ denote the affine variety $V(f_1,\cdots,f_s)=\{\mathbf{x}\in\mathbb{C}^r\mid f_k(\mathbf{x})=0 \text{ for each } 1\leq k\leq s\}$.

Let $\Sigma(1)=\{\rho_1,\cdots,\rho_r\}$ denote the set of all one dimensional cones (or called *a ray*) in a fan Σ, and let $\mathbf{n}_k\in\mathbb{Z}^n$ denote the generator of $\rho_k\cap\mathbb{Z}^n$ called *the primitive element* of ρ_k for each $1\leq k\leq r$. Define the affine variety $Z_\Sigma\subset\mathbb{C}^r$ by $Z_\Sigma=V(z^{\hat{\sigma}}|\sigma\in\Sigma)$, where $z^{\hat{\sigma}}$ denotes the monomial given by $z^{\hat{\sigma}}=\prod_{1\leq k\leq r,\mathbf{n}_k\notin\sigma}z_k\in\mathbb{Z}[z_1,\cdots,z_r]$ $(\sigma\in\Sigma)$. Let $G_\Sigma\subset\mathbb{T}^r$ denote the subgroup consisting of all r-tuples $(\mu_1,\cdots,\mu_r)\in\mathbb{T}^r$ such that $\prod_{k=1}^r\mu_k^{\langle\mathbf{m},\mathbf{n}_k\rangle}=$

1 for any $\mathbf{m} \in \mathbb{Z}^n$, where we set $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k$ for $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$. We say that a set of primitive elements $\{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_s}\}$ is *primitive* if they do not lie in any cone in Σ but every proper subset does. It is known that

$$Z_\Sigma = \bigcup_{\{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_s}\}:\ \text{primitive}} V(z_{i_1}, \cdots, z_{i_s}).$$

Note that Z_Σ is a closed variety of dimension $2(r - r_{\min})$, where we set

$$r_{\min} = \min \big\{ s \in \mathbb{Z}_{\geq 1} \mid \{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_s}\} \text{ is primitive} \big\}.$$

It is also known that if the set $\{\mathbf{n}_1, \cdots, \mathbf{n}_r\}$ spans $\mathbb{R}^n$, there is an isomorphism $X_\Sigma \cong (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma$, where the group G_Σ acts on the complement $\mathbb{C}^r \setminus Z_\Sigma$ by the coordinate-wise multiplication.

For connected spaces X and Y, let $\mathrm{Map}(X, Y)$ be the space of all continuous maps $f : X \to Y$, and let $\mathrm{Map}^*(X, Y)$ denote the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, X)$, let $F(\mathbb{R}\mathrm{P}^m, X; g)$ denote the subspace of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X)$ given by

$$F(\mathbb{R}\mathrm{P}^m, X; g) = \{ f \in \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X) : f|\mathbb{R}\mathrm{P}^{m-1} = g \},$$

where we identify $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{R}\mathrm{P}^m, X; g) \simeq \Omega^m X$.

From now on, we assume that the following two conditions are satisfied:

(1.1) Let Σ be a fan in $\mathbb{R}^n$, $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ be the set of all one-dimension cones in Σ, and all primitive elements $\{\mathbf{n}_1, \cdots, \mathbf{n}_r\}$ of the fan Σ spans $\mathbb{R}^n$, where $\mathbf{n}_k \in \mathbb{Z}^n$ denotes the primitive element of ρ_k for $1 \leq k \leq r$.

(1.2) Let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r-tuple of integers such that $\sum_{k=1}^{r} d_k \mathbf{n}_k = \mathbf{0}$.

Then, we can identify $X_\Sigma = (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma$ as above. For each $(a_1, \cdots, a_r) \in \mathbb{C}^r \setminus Z_\Sigma$, we denote by $[a_1, \cdots, a_r]$ the corresponding element of X_Σ. Let $\mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \ldots, z_m]$ denote the subspace consisting of all homogeneous polynomials of degree d. Let $A_D(m)$ denote the space

$$A_D(m) = \mathcal{H}_{d_1,m} \times \mathcal{H}_{d_2,m} \times \cdots \times \mathcal{H}_{d_r,m}$$

and let $A_{D,\Sigma}(m) \subset A_D(m)$ denote the subspace consisting of all r-tuples $(f_1, \cdots, f_r) \in A_D(m)$ such that $(f_1(\mathbf{x}), \cdots, f_r(\mathbf{x})) \notin Z_\Sigma$ for any $x \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$. Let $x_0 \in X_\Sigma$ be the base point such that $x_0 = [x_{1,0}, \cdots, x_{r,0}]$ for some fixed $(x_{1,0}, \cdots, x_{r,0}) \in \mathbb{C}^r \setminus Z_\Sigma$. Then let $A_D(m, X_\Sigma) \subset A_{D,\Sigma}(m)$ denote the subspace consisting of all r-tuples $(f_1,\cdots,f_r)\in A_{D,\Sigma}(m)$ satisfying the condition $(f_1(\mathbf{e}_1),\cdots,f_r(\mathbf{e}_1))=(x_{1,0},\cdots,x_{r,0})$, where $\mathbf{e}_1=(1,0,\cdots,0)\in\mathbb{R}^{m+1}$, and let us choose $[\mathbf{e}_1]=[1:0:\cdots:0]$ as the base-point of $\mathbb{RP}^m$. Define the natural map $j'_D:A_{D,\Sigma}(m)\to\mathrm{Map}(\mathbb{RP}^m,X_\Sigma)$ by

$$j'_D(f_1,\cdots,f_r)([x_0:\cdots:x_m])=[f_1(\mathbf{x}),\cdots,f_r(\mathbf{x})]$$

for $\mathbf{x}=(x_0,\cdots,x_m)\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}$. Since the space $A_{D,\Sigma}(m)$ is connected, the image of j'_D lies in a connected component of $\mathrm{Map}(\mathbb{RP}^m,X_\Sigma)$, which is denoted by $\mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma)$.

This also gives the natural map $j'_D:A_{D,\Sigma}(m)\to\mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma)$. Note that $j'_D(f_1,\cdots,f_r)\in\mathrm{Map}^*(\mathbb{RP}^m,X_\Sigma)$ if $(f_1,\cdots,f_r)\in A_D(m,X_\Sigma)$. Hence, if we set $\mathrm{Map}^*_D(\mathbb{RP}^m,X_\Sigma)=\mathrm{Map}^*(\mathbb{RP}^m,X_\Sigma)\cap\mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma)$, we have the natural map $i_D=j'_D|A_D(m,X_\Sigma):A_D(m,X_\Sigma)\to\mathrm{Map}^*_D(\mathbb{RP}^m,X_\Sigma)$.

Suppose that $m\geq 2$ and let us choose a fixed element $(g_1,\cdots,g_r)\in A_D(m-1,X_\Sigma)$. For each $1\leq k\leq r$, let $B_k=\{g_k+z_mh:h\in\mathcal{H}_{d_k-1,m}\}$. Then define the subspace $A_D(m,X_\Sigma;g)\subset A_D(m,X_\Sigma)$ by

$$A_D(m,X_\Sigma;g)=A_D(m,X_\Sigma)\cap(B_1\times B_2\times\cdots\times B_r).$$

It is easy to see that $i_D(f_1,\cdots,f_r)|\mathbb{RP}^{m-1}=g$ if $(f_1,\cdots,f_r)\in A_D(m,X_\Sigma;g)$, where g denotes the map in $\mathrm{Map}^*_D(\mathbb{RP}^{m-1},X_\Sigma)$ given by

$$g([x_0:\cdots:x_{m-1}])=[g_1(\mathbf{x}),\cdots,g_r(\mathbf{x})]\quad\text{for }\mathbf{x}=(x_0,\cdots,x_{m-1})\in\mathbb{R}^m\setminus\{\mathbf{0}\}.$$

Then, define the map $i'_D:A_D(m,X_\Sigma;g)\to F(\mathbb{RP}^m,X_\Sigma;g)\simeq\Omega^mX_\Sigma$ by the restriction $i'_D=i_D|A_D(m,X_\Sigma;g)$. Now define the equivalence relation "$\sim$"on $A_{D,\Sigma}(m)$ by $(f_1,\cdots,f_r)\sim(g_1,\cdots,g_r)$ if there exists some element $\lambda\in\mathbb{R}^*$ such that $f_k=\lambda^{d_k}g_k$ for any $1\leq k\leq r$. We denote by $\widetilde{A_D}(m,X_\Sigma)$ the quotient space $\widetilde{A_D}(m,X_\Sigma)=A_{D,\Sigma}(m)/\sim$. Then define the map $j_D:\widetilde{A_D}(m,X_\Sigma)\to\mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma)$ by $j_D([f_1,\cdots,f_r])([x_0,\cdots,x_r])=[f_1(\mathbf{x}),\cdots,f_r(\mathbf{x})]$ for $\mathbf{x}=(x_0,\cdots,x_m)\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}$.

A map $f:\mathbb{RP}^m\to X_\Sigma$ is called *an algebraic map of degree D* if it can be represented as a rational map (or regular map) of the form

$$f=j'_D(f_1,\cdots,f_r)=[f_1,\cdots,f_r]\quad\text{for some }(f_1,\cdots,f_r)\in A_{D,\Sigma}(m).$$

We denote by $\mathrm{Alg}_D(\mathbb{RP}^m,X_\Sigma)$ the space of all algebraic maps $f:\mathbb{RP}^m\to X_\Sigma$ of degree D. Consider the natural projection $\Gamma'_D:A_{D,\Sigma}(m)\to\mathrm{Alg}_D(\mathbb{RP}^m,X_\Sigma)$ given by $\Gamma'_D(f_1,\cdots,f_r)=j'_D(f_1,\cdots,f_r)=[f_1,\cdots,f_r]$. Then it clearly induces a natural projection $\Gamma_D:\widetilde{A_D}(m,X_\Sigma)\to\mathrm{Alg}_D(\mathbb{RP}^m,X_\Sigma)$.

For $g \in \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$, let $\mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ and $\mathrm{Alg}^*(\mathbb{R}\mathrm{P}^m, X_\Sigma; g)$ denote the subspaces of $\mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma)$ given by

$$\begin{cases} \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) & = \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \cap \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) & = \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \cap F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \end{cases}$$

Then the projection Γ'_D induces the projection maps by the restrictions

$$\begin{cases} \Psi_D : A_D(m, X_\Sigma) \to \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ \Psi'_D : A_D(m, X_\Sigma; g) \to \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \end{cases}$$

Let

$$\begin{cases} j_{D,\mathbb{C}} : \mathrm{Alg}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \hookrightarrow \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i_{D,\mathbb{C}} : \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \hookrightarrow \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i'_{D,\mathbb{C}} : \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \hookrightarrow F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

denote the inclusions. It is easy to see that the following equalities hold:

$$\begin{cases} j_D = j_{D,\mathbb{C}} \circ \Gamma_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i_D = i_{D,\mathbb{C}} \circ \Psi_D : A_D(m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i'_D = i'_{D,\mathbb{C}} \circ \Psi'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

Let $g \in \mathrm{Alg}^*_D(\mathbb{R}\mathrm{P}^{m-1}, X_\Sigma)$ be any fixed algebraic map of degree D and we choose an element $(g_1, \cdots, g_r) \in A_D(m-1, X_\Sigma)$ such that $g = [g_1, \cdots, g_r]$.

Now we can state the our main result as follows.

Theorem 1.1 ([14]). *Let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ and let Σ be a complete smooth fan in $\mathbb{R}^n$ satisfying the above conditions (1.1) and (1.2). Then if $2 \leq m \leq 2(r_{\min} - 1)$ and X_Σ is a smooth compact toric variety, the maps*

$$\begin{cases} j_D : \widetilde{A_D}(m, X_\Sigma) \to \mathrm{Map}_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i_D : A_D(m, X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m, X_\Sigma) \\ i'_D : A_D(m, X_\Sigma; g) \to F(\mathbb{R}\mathrm{P}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

are homology equivalences through dimension $D(d_1, \cdots, d_r; m)$, where the number $D(d_1, \cdots, d_r; m)$ is given by

$$D(d_1, \cdots, d_r; m) = (2r_{\min} - m - 1)\min\{d_1, d_2, \cdots, d_r\} - 2. \quad \Box$$

Remark. A map $f : X \to Y$ is called *a homology equivalence through dimension N* if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq N$.

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